Degree Thesis

Flexural Rigidity (D) in Beams

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Abstract:

This thesis presents the theory behind the symmetrical and unsymmetrical beams with different cross-sections, the mathematical procedure in calculating the flexural rigidity of symmetrical beams and summarizing the experimental verification by mathematical data processing of the flexural rigidity by three-point bending.

The core of the method section is to test theoretically by using the composite compressive strength modeler (CCSM) software and experimentally in the laboratory by using the material bending machine a solid fiberglass and a sandwich beam. The results obtained for the solid fiberglass was found to be theoretically 32 Nmm^2 and experimentally 31.1 $MNmm^2$. For the sandwich beam along the direction of the orientation of the fiber theoretically 155 Nmm^2 and experimentally 153.3 Nmm^2. As for the sandwich beam with the same properties but with a direction perpendicular to the fiber theoretically 45.1 Nmm^2 and experimentally 47.22 Nmm^2. The comparison of flexural rigidity values was found to be 3.1% for the solid fiberglass, 1.1% for the sandwich beam along the direction of the orientation of the fiber and -4.6% with a direction perpendicular to the fiber. Moreover, for the four-section module, the values obtained for the moments of the outer layer contributed seven times more to rigidity than the inner layer and for the six-section module the second outermost layer contributed seven times more than the innermost layer and the outermost layer nineteen times more to rigidity than the innermost layer.

Keywords:	Flexural rigidity, deflection in beams, symmetrical bending, sandwich structure, composite beams, (CCSM) modeler, three-point bending
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List of Symbols

Name	Symbol	Unit
1. Flexural rigidity	D	Nm^2
2. Bending modulus (material property)	E	Pa = N/m^2
3. Young´s modulus of faces	E_f	Pa
4. Second moment of inertia	I	m^4
5. Width	b	m
6. Thickness	t	m
7. Distance	d	m
8. Force	F	N
9. Height	h	m
10. Moment/Bending moment	M	Nm
11. Stress	σ	Pa = N/m^2
12. Length	L	m
13. Area	A	m^2
14. Strain	E	- or mm/mm
15. Shear stress	Q	Pa
16. Slope	Θ (k)	N/mm
17. Distance along beam	x	m
18. Radius of neutal axis	r	m
19. Distance of surface (NS)	y	m
20. Torque	τ	Nm
21. Deflection	Y	mm
22. Distributed load	W	N/m

FOREWORD

I would like to thank Mr. Rene Herrmann for the supervision throughout this thesis.

I would like to thank all the professors and staff for their guidance during my three years of studying at Arcada University of Applied Sciences. Special thanks to Mr. Erland Nyroth for his encouragement and advices throughout my studies.

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1 INTRODUCTION

The purpose of this thesis work is to summarize the mathematical procedure to calculate the bending stiffness of symmetrical and unsymmetrical beams with different cross sections such as I-, Rectangular-, and T-cross-sections.

This paved the way for the summary of the experimental verification by mathematical data processing of the flexural rigidity by 3-point, a review on composites and metals as well the comparison between them.

Moreover, it is intended to summarize on some examples the theoretical and experimental flexural rigidity and to review international measurement standards for polymer and composite materials, finally, using the CCSM (Composite Compressive Strength Modeler) for deformation analysis of composite materials.

1.1 Problem Definition

Structural efficiency is one of the most important character to engineers in the aerospace, boats, car industries and in many other fields. Therefore, a high-performance product made of lightweight materials and yet efficient to withstanding harsh loading conditions is required.

The stiffness as a material property is very important in structural engineering as it extents the materials resist deformation in response to applied load. The use of sandwich composites has been considered in this thesis due to the relevance of its stiff light weight structure, which significantly increases the load resistance capacity on one hand without much increase in weight on the other hand.

To design a member such as a beam, it is very essential to make sure that it satisfies specific strength, stability, and deflection requirements. Therefore, the bending stiffness analysis plays a major role in choosing materials as it shows how a component behaves under certain loads.

1.2 Aims and Objectives

In order to fulfil the aim of this thesis, the research is conducted aiming to calculating mathematically the flexural rigidity of symmetrical cross sections, verify the experimental results of mathematical data processing of the flexural rigidity by 3-point bending, present some examples to illustrate the theoretical and experimental flexural rigidity, review of verification methods for 4-point bending, comparing composite materials to metals using some international measurement standards and using composite compressive strength modeler (CCSM) software for modelling of flexural rigidity.

1.3 Method

In order to achieve the aim of this thesis and get the theoretical knowledge of the problem, the theory of bending stiffness for different beams and plates have been studied. In the consideration of the bending stiffness analysis of composite material structures, the composite compressive strength modeler (CCSM) and experiments on different materials in the laboratory have been conducted.

There are several software packages available for rigidity analysis and deformation tests. In this thesis work, the composite compressive strength modeler (CCSM) software is used for flexural rigidity simulation.

1.4 Background

Flexural rigidity is related to bending and non-rigid structures as a force couple that is required to bend a non-rigid structure in one unit of curvature, where force couples also known as pure moments are the forces that rotate a body without translation or acceleration of the center of mass. [1]

Since force couples are free vectors in rigid bodies, their effects on a body are independent of the point of application. A couple is known as a combination of two forces, they are equal in magnitude, asymmetrically directed, and displaced by a perpendicular distance or moment. [1]

1.4.1 Flexural Rigidity

Flexural rigidity can be modelled by using different methods. In this scientific discourse, three of the very well-known methods in the bending stiffness of a profile are treated.

The first method can be applied when the modulus is constant, and the cross section is simple then the single formula of second moment of inertia (equation 2, p.12) can be used to obtain the required results of flexural rigidity D. [2]

The second method is used when the modulus is constant, and the cross section is not simple but can be however reduced to a simple cross section with a displacement from the neutral axis, in this case the parallel axis theorem can be used effectively by determining the second moment of area or mass moment of inertia of a rigid body about any axis. [3]

While the third method which is known as the sandwich beam theory or the Euler-Bernoulli beam theory that describes the behavior of a beam, plate, or a shell when calculating the load-carrying and deflection characteristics of beams. It is used when the profile is not constant in bending modulus, but the profile can be approximated or considered as a layered structure of different materials. [4]

The flexural rigidity (D) is defined as EI. In a beam or rod, flexural rigidity varies along the length as a function of (x):

$$EI\frac{dy}{dx} = \int_0^x M(x)dx + C_1 \qquad (1)\ [1]$$

Where,

E = Young´s modulus (Pa)
I = second moment of area (m^4)
y = transverse displacement at x (m)
M (x) = bending moment at x (Nm)

2 THEORY

The Euler Bernoulli´s Equation and the bending stiffness of a rigid body expresses clearly the flexural rigidity. [1] Euler Bernoulli is known as an engineer of beam theory, which simplifies the linear theory of elasticity. It provides a calculation for the load-carrying and deflection characteristics of beams and it covers the case for infinitesimal strains and small rotations of a beam which are only subjected to lateral loads. [5]

The theory can be extended in a straightforward manner to problems with moderately large rotations by using the von Kármán strains, which provides that the strain remains small. However, in this thesis work the focus is on bending stiffness. [5]

Stiffness describes the rigidity of an object, in other words it is the resistance to deformation when a force is applied on it. So, the flexibility of materials is very essential, the more flexible an object is, the less stiff it is. The stiffness of a body is a measure of the resistance offered by an elastic body to deformation.

The bending stiffness of a beam is a function of elastic modulus “E”, the area moment of inertia “I “of the beam cross-section about the axis of interest, length of the beam and the beam boundary conditions. The bending stiffness of a beam can be analytically derived from the equation of beam deflection when it is applied by a force. [1]

2.1 Second Moment of Area

Second moment of area is the property of a cross section. It is normally used to predict the resistance of beams to bending and deflection. The deflection of beams under a certain load doesn´t depend only on the load but also on the geometry of the beam´s cross section.

Beams with a large second moment of area are stiffer than those with a smaller second moment and therefore, they are more resistant to bending. The second moment of area has in other words the meaning of second moment of inertia: [2]

$$\mathrm{I} = \mathrm{I}_0 + \mathrm{A}\mathrm{d}^2 \qquad (2)\ [2]$$

Where,

I = second moment of inertia (m^4)

I_0 = second moment of area at centroidal axis (m^4)

A = area (m^2)

d = distance (m)

2.2 Materials

Nowadays, the designing of materials and material properties play a big role in the development of materials. However, there are two main types of materials. The first type is known as structural materials, where the mechanical properties are mainly considered, such as strength, stiffness, and deformation. And the second type is known as functional materials where, the magnetic properties are mainly considered, such as the sound, light, electricity, and heat. [6]

Moreover, these materials can be divided into three types depending on the way the atoms or molecules are bonded together. Metallic materials describe the first type, where metal elements are combined with metal bonds, organic polymer materials stand for the second type, where non-metallic elements are bonded covalently to macromolecular compounds while ceramic materials describe the third type, where non-metallic elements and metal elements are combined by covalent bonds, ionic bonds, or a mixture of the two bonds. [6]

2.2.1 Composite Materials

Composite material also known as a multi-phase combination material. Composites are formed by combining materials together to form an overall structure with properties that differ from the sum of the individual components. [7]
They are generally used for buildings, bridges, and structures due to their benefits such as chemical and corrosion resistance, durable, flexible in design, high flexural modulus to carry demanding loads, high impact strength, high performance at elevated temperature, etc. [7]

Composite materials consist of matrix material and reinforcing material as shown in figure 2.1 below, where matrix materials are defined as a continuous phase, which includes metal matrix composite materials, inorganic non-metallic matrix composite materials as well polymer matrix composites by the different matrix material. Reinforcing material is defined as a dispersed phase which includes usually fibrous materials, such as glass fiber, and organic fiber. [6]

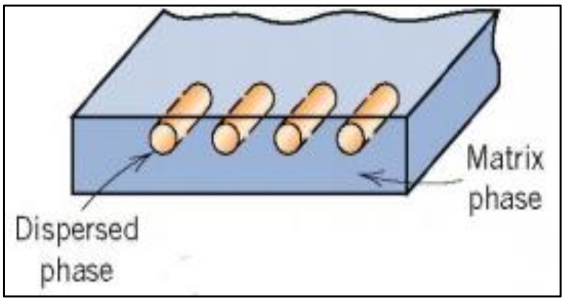

Figure 2.1 Composite structure [6]

Normally the strength of fiber depends on its length and orientation with respect to the stress direction, however, the strength and modulus of fiber are much higher than the matrix material, due its length and accordingly the orientation of stress direction therefore, fibers are the main load-bearing components. However, to firmly bond fibers together, there must be a matrix material with good adhesion properties that can provide a uniformly distributed applied load and transfer the loads to fiber. [6]

Composite materials should have at least four characteristics. The composite should be made of a non-homogeneous material, the components that make up the composite should have different levels of performance, the performance should be the main characteristic of all composites, and the fraction of each component of the composites should be larger than 10 percent of the volume of the composite. [6]

2.2.2 Composites versus Metallic

Composites and metals have different physical characteristics. For instance, composites are greatly anisotropic which means their properties and values are changeable with direction as well as their strength and stiffness, depending on the orientation of the reinforcing fibers. [8]

In addition, composites are lighter in weight, they can tailor the lay-up for optimum strength and stiffness, improve fatigue life, corrosion resistance, etc. While the metals are isotropic which means their physical properties have the same values in different directions, they are much heavier than composites in weight. Figure 2.2 and table 1 below illustrate an example of some main differences between a composite and a metal. [8]

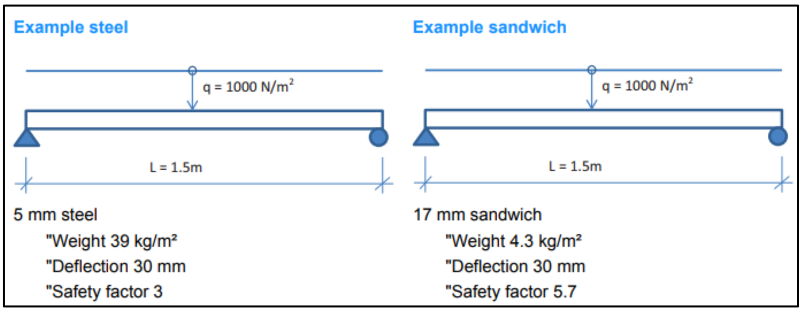

Figure 2.2 Steel vs. sandwich [9]

Table 1. Composites versus metals under certain conditions. [9]

Condition	**Comparative behavior relative to metals**
Load-strain relationship	More linear strain to failure
Static	Greater sensitivity
Fatigue	Less sensitivity
Transverse properties	Weaker
Mechanical property	Higher
Fatigue strength	Higher
Sensitivity to hydrothermal	Greater
Sensitivity to corrosion	Much less
Damage growth mechanism	In-plane delamination instead of through thickness cracks

2.3 Bending Stiffness of Beams

This section covers some concepts about the bending stiffness of symmetrical and unsymmetrical beams with different cross sections, the study of simple bending, and bending of composite beams.

2.3.1 Theory of Simple Bending with Assumptions Made

Simple bending happens at the length of a beam that is subjected to a constant bending moment and when there is no shear force, that is ($Q = 0$). Which means that the stresses will be arranged along the length of the beam, due to the bending moments only. Therefore, that length of the beam is said to be in pure bending.

The assumptions made in the theory of simple bending are known as, the beam is homogeneous and isotropic, the Young´s modulus of elasticity in tension and compression has the same value, the transverse sections which were plane before bending remain plane after bending, the beam at the beginning of the experiment is straight, and all longitudinal filaments bend into circular arcs with a common center of curvature, the radius of curvature is large compared with the dimensions of the cross section, and each layer of the beam is free to expand or contract, independently of the layer above or below it. [10]

However, there are some other tools to use for complicated designs such as a superposition principle or property tool. This tool is used in linear systems for the beams. The superposition principle is one of the most important tools for solving beam loading problems as it allows the simplification of very complicated designs. [10]

For the beams that are subjected to several loads of different types the resulting shear force, bending moment, slope and deflection can be found at any location by summing the effects due to each load acting separately to the other loads. The superposition principle is a combination of homogeneity and additivity and satisfies the following equations. [10]

Additivity:

$$F(x_1 + x_2) = F(x_1) + F(x_2) \qquad (3)\ [10]$$

Homogeneity:

$$F(ax) = aF(x) \qquad (4)\ [10]$$

Simple bending is when a straight bar of homogeneous material is subjected to only a moment at one end and an equal and opposite moment at the other end.

2.3.2 Bending of Composite Beams

A composite beam is made of two or more different materials where these materials are connected rigidly, and the composite behaves like a single piece once it is made. The basic assumption in this case assumes that the plane surface remains plane during bending within the elastic limit [10]. Therefore, the strain remains constant down at the full width of the beam, where the deflection is proportional to the distance from the neutral axis of the beam, in this case the strain can be found easily, and it is equal to (stress/Young´s Modulus E) [11]. As shown in figure 2.3 below.

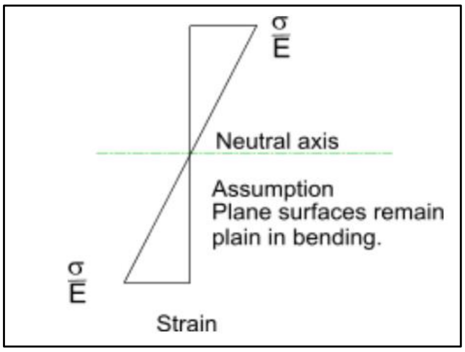

Figure 2.3 Strain [11]

The dimensions of the replacement material have the same mechanical properties as the original material, and the overall depth of the transformed section is the same as the original section. Therefore, the resulting strain in any element dA of the transformed section must be constant. [11]

$$\text{Strain } (\varepsilon) = \frac{\sigma}{E} \tag{5}$$ [11]

Where,

ε = strain (mm/mm)

σ = stress (N/m^2)

E = bending modulus (material property) (Pa)

The bending stresses of a composite beam can be calculated using two conditions as follows. [10]

1. The resulted strain on a layer with an equal distance from the neutral axis is the same for both materials.

$$\frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \tag{6}$$ [10]

$$\sigma_1 = \sigma_2 \cdot \left(\frac{E_1}{E_2}\right) = m\sigma_2$$

2. The moment of resistance of a composite beam can be determined simply by summing up the individual moments of resistance of the members.

$$M = M_1 + M_2 \tag{7}$$ [10]

$$M_1 = \sigma_1 \cdot \left(\frac{I_1}{y}\right)$$

$$M_2 = \sigma_2 \cdot \left(\frac{I_2}{y}\right)$$

$$M = \frac{\sigma_1}{y} I_1 + \frac{\sigma_2}{y} I_2$$

Where,

σ = stress (N/m^2)

E = bending modulus (material property) (Pa)

M = bending moment (Nm)

I = moment of inertia (m^4)

y = distance (m)

m = modular ratio, which means in construction the ratio of Young's Moduli of elasticity of the two different materials. (N/m^2)

2.3.3 Symmetrical Bending

The symmetrical bending can be defined as a bending where the bending moments are symmetrical around the neutral axis that passes through the center. Symmetrical bending mainly happens in beams that have symmetrical cross section which can be either single or double layers. [11]

The second moment of area which is needed in calculations of bending in beams and also known as the moment of inertia of a shape, it depends on the geometry of objects and it describes how points or particles of an object or an area are distributed about an axis that can be chosen arbitrarily. [11]

2.3.4 Examples of Symmetrical Bending

One of the examples of symmetrical bending is an I-cross-section beam. Tension and compression loads are applied to the flanges and the body (web) is the core that keeps the facings in place by resisting transverse shear loads. [12]

The second moment of area is very high in I-section beam because most of its material is located in the flanges which are placed far from the center of bending (neutral axis), and the web has enough material that makes the flanges work together and resist shear and buckling. [12]

There are different ways to calculate the bending stiffness of beams. However, for the following shape as an example and since the beam is symmetric from the top to bottom, there is no need to find the centroid of the location. [13]

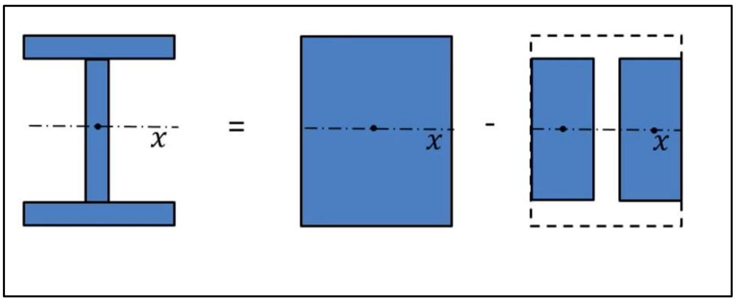

Figure 2.4 I-beam [13]

As it shows from the figure 2.3 above, the I cross section consists of three pieces, instead of treating each piece separately and making separate calculations. It is valuable to treat the whole beam as a rectangle and as a total solid cross section which can be divided into two symmetrical pieces around x-axis and since the centroids of all segments lie on x-axis, there is no need to use the parallel axis theorem. [13]

So, the solution for this particular example can be obtained using the following simple moment equation.

$$I_x = \frac{1}{12} b_{center} h^3{}_{center} \qquad (8)\ [13]$$

The flexural rigidity can be determined from equation 1 page 13.

However, in the case of taking all three pieces into consideration, knowing that the beam is symmetric and the center of gravity in the middle then the flexural rigidity D is: [1]

$$D = \sum D_i = E\left(I_{top} + I_{bottom} + I_{center}\right) \qquad (9)$$

Considering that, [13]

$$I_{center} = \frac{1}{12} b_{center} h^3_{center}$$

And for the bottom and top parts, there is a distance d from the neutral line, so the moment is: [3]

$$I = I_{center} + Ad^2$$

$$I_{top} = \frac{1}{12} b_{top} h_{top}^3 + b_{top} h_{top} d^2$$

Where the displacement d is:

$$d = \frac{h_{center}}{2} + \frac{h_{top}}{2}$$

And the area A is:

$$A = b_{top} h_{top}$$

Since it is symmetric then:

$$I_{top} = I_{bottom}$$

Another well-known example of a symmetrical bending is a rectangular cross section beam. The calculation in this case can be done in different ways, one way is by finding the second moment of inertia of the area of a structural section and using the parallel axis theorem. Another common way used in calculations is known as section modulus method. An explanation of both methods is discussed below. [12]

1. Parallel Axis Theorem:

In this method a rectangular cross section beam that has a width b and height h is considered as shown in figure 2.4 below.

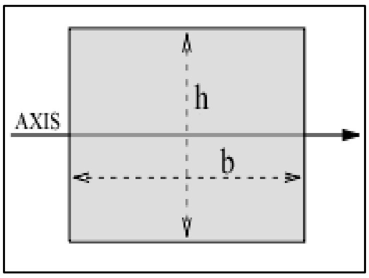

Figure 2.5 Rectangular cross-section

Then the area A is:

$$A = b \cdot h \qquad (10)\ [12]$$

And the moment of inertia I around x-axis that passes through the centroid of the rectangle is:

$$I = \frac{1}{12} bh^3$$

And it can be used to calculate the flexural rigidity as follows: [12]

$$D = E \cdot I = E \cdot \frac{1}{12} bh^3$$

When the rotation is around an axis at a distance d from x-axis and is parallel to the centroidal axis as shown in figure 2.6 below.

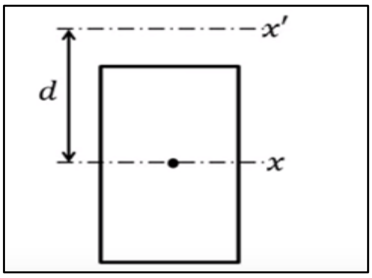

Figure 2.6 Centroidal x-axis

Then we use the following parallel axis theorem: [3]

$$I_{x'} = I_x + Ad^2$$

2. Section Modulus Method:

In this method the cross-section beam is considered to be symmetric from top to bottom, so there is no need to find location centroid. Therefore, the entire rectangle is taken as a total solid cross section and divided into many sections. However, in this thesis work two-, four-, and six sections are considered below. [13]

❖ Two-Section Modulus:

In this case a rectangular beam is cut into two-section modulus as in figure below.

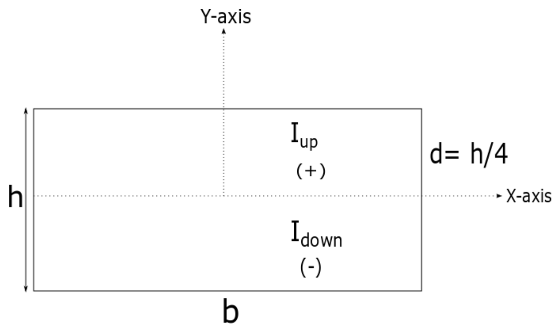

Figure 2.7 Two-sections

The moment of inertia for each element from top to bottom in this case can be obtained first and then summing up to get the main moment I which can be used to calculate the flexural rigidity as below:

For each element – top or bottom: [3]

$$I = I_0 + Ad^2$$

Where,

$$A = b \cdot \frac{h}{2}$$

$$d = \frac{h}{4}$$

$$I_0 = \frac{1}{12} b \cdot \left(\frac{h}{2}\right)^3$$

b = width (m)

h = height (m)

The moment of inertia for the upper part of the beam is found by substituting A and r into moment equation so we get:

$$I_{up} = \frac{1}{12} b \left(\frac{h}{2}\right)^3 + b\left(\frac{h}{2}\right) \cdot \left(\frac{h}{4}\right)^2$$

$$I_{up} = \frac{1}{96} bh^3 + bh^3 \frac{1}{32}$$

And the moment of inertia for the bottom part is determined by substituting the same A and r values into the moment equation again, taking into consideration the negative sign of d, that is (-d). So, we get:

$$I_{down} = \frac{1}{12} b \left(\frac{h}{2}\right)^3 + b\left(\frac{h}{2}\right) \cdot \left(-\frac{h}{4}\right)^2$$

$$I_{down} = \frac{1}{96} bh^3 + bh^3 \frac{1}{32}$$

Then the total moment I is the sum of top and bottom moments:

$$I_{tot} = I_{up} + I_{down} = 2I = bh^3 \left(\frac{32 + 96}{32 \cdot 96}\right) \cdot 2$$
$$= bh^3 \left(\frac{128}{32 \cdot 96}\right) \cdot 2 = bh^3 \frac{1}{12}$$

So, the flexural rigidities can be calculated when E, b, h are given and as follows:

$$D = E \cdot I_{tot}$$
$$D = E \left(bh^3 \frac{1}{12}\right)$$

❖ Four-Section Modulus:

In this module the beam is divided to four sections, the individual moments are calculated in a similar way to the two-section modulus and then the total moment is substituted in the equation of flexural rigidity. The figure 2.8 below illustrates the sections and dimensions used in calculations.

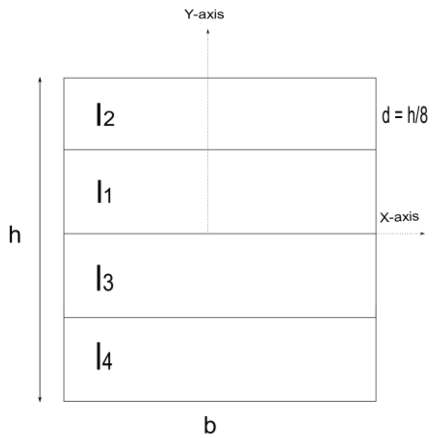

Figure 2.8 Four-sections

For each element – top or bottom: [3]

$$I = I_0 + Ad^2$$

Where,

$$\mathrm{A} = \mathrm{b} \cdot \frac{\mathrm{h}}{4}$$

$$\mathrm{d} = \frac{\mathrm{h}}{8}$$

That leads to,

$$I_0 = \frac{1}{12} b \cdot \left(\frac{h}{2}\right)^3$$

The first moment I_1 can be obtained by substituting these values into the following equation:

$$\mathrm{I_1} = \mathrm{I_0} + \mathrm{Ad^2} = \frac{1}{12}\mathrm{b} \cdot \left(\frac{\mathrm{h}}{2}\right)^3 + \frac{\mathrm{bh}}{4} \cdot \left(\frac{\mathrm{h}}{8}\right)^2$$

$$\mathrm{I_1} = \mathrm{bh^3} \cdot \left(\frac{1}{12} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{1}{64}\right)$$

$$= \mathrm{bh^3}\left(\frac{1}{96} + \frac{1}{256}\right)$$

For $\mathrm{I_2}$ an additional distance of $\left(\frac{\mathbf{h}}{\mathbf{4}}\right)$ which is $\left(\frac{\mathbf{h}}{\mathbf{8}}\right) + \left(\frac{\mathbf{h}}{\mathbf{8}}\right)$ is considered in calculations as shown below:

$$I_2 = \frac{1}{12} b \cdot \left(\frac{h}{4}\right)^3 + \frac{bh}{4} \cdot \left(\frac{h}{4} + \frac{h}{8}\right)^2 = \frac{1}{12} b \frac{h^3}{64} + b \frac{h}{4}\left(\frac{3h}{8}\right)^2$$

$$\mathrm{I_2} = \mathrm{bh^3}\left(\frac{1}{768} + \frac{9}{256}\right)$$

The moments I_3 and I_4 can be obtained in the same way and then I_{total} which is the sum of the moments of all elements is substituted in the flexural rigidity equation 1 page 13.

Moreover, the moments related to the thickness t can be calculated in a similar way.

The general formula for the moment I in this case is:

$$\mathrm{I} = \frac{1}{12}\mathrm{b}(4\mathrm{t})^3 = \frac{\mathrm{bt^3}}{12} \cdot 64 \qquad (11)\ [3]$$

So, for $\mathrm{I_1}$ with thickness t_1 we get:

$$I_1 = \frac{1}{12} b t_1^3 + bt \cdot \left(\frac{t}{2}\right)^2$$

And for I_2 we get,

$$I_2 = \frac{1}{12} b t_2^3 + bt \cdot \left(-\frac{t}{2}\right)^2$$

For the first and the second moment $I_{1/2}$ we get,

$$I_{1/2} = b\left(\frac{1}{12} t^3 + t \cdot \left(\frac{t}{2}\right)^2\right) \cdot 2$$

$$I_{1/2} = bt^3 \left(\frac{1}{12} + \frac{1}{4}\right) \cdot 2 = bt^3 \left(\frac{4+12}{48}\right) \cdot 2$$

$$I_{1/2} = bt^3 \frac{16}{24}$$

In a similar way the third and the fourth moments are obtained and then we get $I_{3/4}$:

$$I_{3/4} = b\left(\frac{1}{12} t^3 + t \cdot \left(\frac{3}{2} t\right)^2\right) \cdot 2$$

$$I_{3/4} = bt^3 \left(\frac{1}{12} + \frac{9}{4}\right) \cdot 2 = bt^3 \left(\frac{4 + (9 \cdot 12)}{48}\right) \cdot 2$$

$$I_{3/4} = bt^3 \frac{224}{48} = bt^3 \frac{112}{24}$$

So, I_{total} is:

$$I_{tot} = \sum I_i = I_{1/2} + I_{3/4} = \frac{bt^3}{24}(128)$$

$$I_{tot} = \frac{bt^3}{12} \cdot 64$$

It can be seen from the values obtained for the moments of the outer and the inner layers, that is values of $I_{1/2}$ and $I_{3/4}$ that the outer layer contributes seven times more to the rigidity than the inner layer, since the ratio of the outer and the inner layer is:

$$\frac{112}{16} = 7$$

❖ Six-Section Modulus:

In this symmetric six section module, the moments for three layers will be considered and calculated and then generalize the results to the other three since the module is symmetric as shown below.

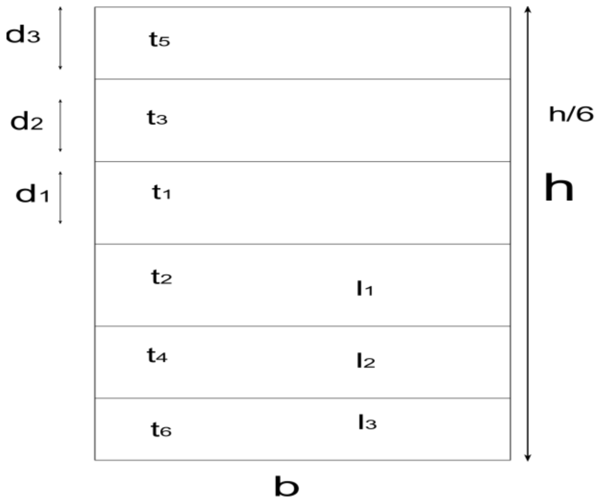

Figure 2.9 Six-sections

For each element – top or bottom:

$$\mathrm{I} = \mathrm{I}_0 + \mathrm{Ad}^2$$

Where,

$$\mathrm{A} = \mathrm{b}\left(\frac{\mathrm{h}}{6}\right)$$

And the three distances are:

$$\mathrm{d}_1 = \frac{\mathrm{h}}{6} \cdot \frac{1}{2}$$

$$\mathrm{d}_2 = \frac{\mathrm{h}}{6} + \frac{\mathrm{h}}{12}$$

$$\mathrm{d}_3 = \frac{\mathrm{h}}{6} \cdot 2 + \frac{\mathrm{h}}{12}$$

Applying the general moment equation below:

$$\mathrm{I} = \frac{1}{12}\mathrm{b}\left(\frac{\mathrm{h}}{6}\right)^3 + \mathrm{b}\frac{\mathrm{h}}{6} \cdot \left(\frac{\mathrm{h}}{12}\right)^2 \qquad (12)\ [2]$$

Then we get I_1:

$$I_1 = bh^3 \left(\frac{1}{12} \cdot \left(\frac{1}{6}\right)^3 + \frac{1}{6} \cdot \left(\frac{1}{12}\right)^2\right)$$

$$= bh^3 \left(\frac{1}{2592} + \frac{1}{864}\right) = bh^3 \frac{1}{648}$$

And for I_2 we get:

$$I_2 = \frac{1}{12} b \left(\frac{h}{6}\right)^3 + b \left(\frac{h}{6}\right) \cdot \left(\frac{3h}{12}\right)^2$$

$$I_2 = bh^3 \left(\frac{1}{12} \cdot \left(\frac{1}{6}\right)^3 + \frac{1}{6} \cdot \left(\frac{1}{4}\right)^2\right)$$

$$= bh^3 \left(\frac{1}{2592} + \frac{1}{96}\right) = bh^3 \frac{7}{648}$$

In the same way I_3:

$$I_3 = \frac{1}{12} b \left(\left(\frac{h}{6}\right)^3 + b \left(\frac{h}{6}\right) \cdot \left(\frac{5}{12} h\right)^2\right)$$

$$I_3 = bh^3 \left(\frac{1}{12} \cdot \left(\frac{1}{6}\right)^3 + \frac{1}{6} \cdot \left(\frac{5}{12}\right)^2\right)$$

$$= bh^3 \left(\frac{1}{2592} + \frac{25}{864}\right) = bh^3 \frac{19}{648}$$

The ratios of I_2 and $I_1$the outer and the inner layers are:

$$\frac{7}{1} = 7$$

The second outermost contributes seven times more to rigidity than to the innermost.

The ratios of I_3 and $I_1$the outer and the inner layers are

$$\frac{19}{1} = 19$$

And the outermost contributes nineteen times more to rigidity than the innermost.

From the results and as a conclusion, it can be seen that the outer moments are also dominating in this case by comparing the obtained values of I_1, I_2, and I_3 which gives a clear idea that the rigidity is higher at outer layers, according to the equation of flexural rigidity.

It should be mentioned that in a rectangular cross section the contact force and deflections are of great importance, especially when the contact force is directly proportional to the area moment of inertia I of the cross section, where the elastic modulus of the beam material and the beam geometry play a big role in the amount of stress and applied force. So, for instance when the beam is pushed down, there will be an upward restoring force which is equal in magnitude but in opposite direction to the applied force as shown in figure 2.11 below. [14]

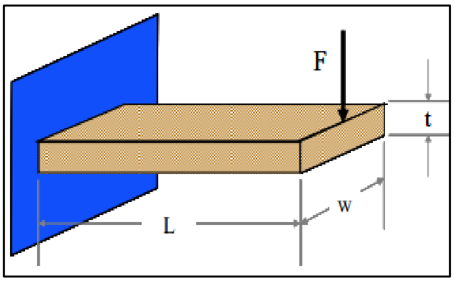

Figure 2.10 Rectangular beam [15]

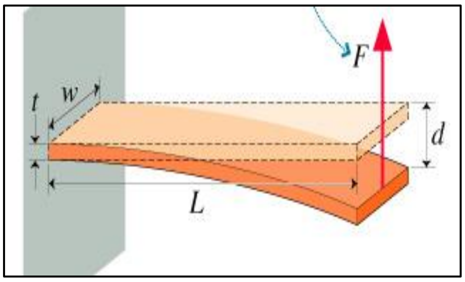

Figure 2.11 Upward restoring force [15]

These forces, deflection, yield, and maximum deflection can be calculated according to the following formulas.

Contact Force:

$$F = \left[\frac{E \cdot w \cdot t}{4 \cdot L^3}\right] \cdot d \qquad (13)\ [14]$$

Deflection at the yield:

$$Y_{yield} = \frac{2 \cdot L^2}{3 \cdot E \cdot t} \cdot \sigma_{yield} \qquad (14)\ [14]$$

Maximum deflection at load:

$$Y = \frac{Fl^3}{48EI} \qquad (15)\ [14]$$

Contact force at yield:

$$F_{yield} = \left[\frac{E \cdot w \cdot t^3}{4 \cdot L^3}\right] \cdot d_{yield} \qquad (16)\ [14]$$

$$= \left[\frac{E \cdot w \cdot t^3}{4 \cdot L^3}\right] \cdot \left[\frac{2 \cdot L^2}{3 \cdot E \cdot t}\right] \cdot \sigma_{yield}$$

Where,

Y = deflection (mm)

d = distance (m)

F = force (N)

σ = stress (N/m^2)

L = length (m)

t = thickness (m)

w = b = width (m)

I = area moment of inertia (m^4)

E = bending modulus (material property) (Pa)

For known parameters such as dimensions, applied force, and other parameters then the maximum deflection at load for instance can be calculated from equation 19 by using flexural rigidity equation. Which shows the importance of flexural rigidity when studying deflection, contact force at yield, deflection at yield, etc.

2.3.5 Unsymmetrical Bending

In the case of nonsymmetrical section, the neutral axis doesn´t pass through the center of the geometrical section. So, the value of *y* which is the distance of the layer from the neutral axis varies for layers that are located for example at the top and bottom of the section. In order to calculate the bending and since the module is unsymmetrical, the center of gravity of the sections must be found first. [10]

The center of gravity (Cg) is defined to be the center to an object´s weight distribution, where the gravity force is considered. It is the point where the object is balanced completely, regardless of point of rotation or turning around. Knowing that the neutral axis normally passes through the center of gravity of the section, so by calculating the center of gravity of the section, one can find the *y* values for topmost layer as well for the bottom layer of the section. [10]

To more explain the unsymmetrical module, a T-cross-section example is studied. The top part of the slab is called flange which resists the compressive stress and the part that lies below the slab is called rib which resists the shear stress. [10]

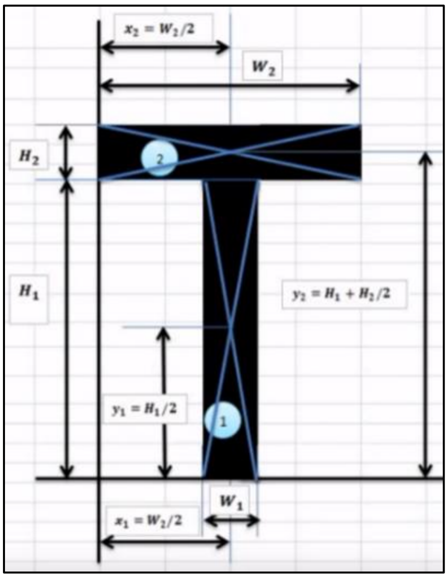

Figure 2.12 T-beam [16]

To find the moment, the centroid $\overline{X}$ and $\overline{y}$ of a T-section are calculated first:

$$\overline{X} = \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2} \quad (17)\ [16]$$

$$\overline{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} \quad (18)\ [16]$$

Where, a_1 and a_2 are the areas of the flange and rib respectively.

The second moment I_{xx} is: [16]

$$I_{xx1} = \frac{1}{12} W_1 {H_1}^3 + a_1 {k_3}^2$$

$$I_{xx2} = \frac{1}{12} W_2 {H_2}^3 + a_2 {k_4}^2$$

$$I_{xx} = I_{xx1} + I_{xx2}$$

Where,

k_i = Radius of gyration

$$k_3 = \bar{y} - y_1$$

$$k_4 = \bar{y} - y_2$$

To calculate I_{yy}: [16]

$$I_{yy1} = \frac{1}{12} W_1 {H_1}^3 + a_1 {k_1}^2$$

$$I_{yy2} = \frac{1}{12} W_2 {H_2}^3 + a_2 {k_2}^2$$

$$I_{yy} = I_{yy1} + I_{yy2}$$

Where,

k_i = Radius of gyration

$$k_1 = \bar{x} - x_1$$

$$k_2 = \bar{x} - x_2$$

So, the flexural rigidity is obtained as in previous sections by substituting the total moment I in the rigidity formula.

2.4 History of Sandwich Structure

According to [17]. Delau in England introduced the first sandwich construction, back to Fairbairn 1849. The first use of sandwich panels "aircraft sandwich" was used in World War II. Mainly because of the lack of other materials during the war.

The Sandwich Structure theoretically appeared in the 50´s. The use of sandwich structure had some limitation in aircraft industry as honeycomb was mainly used as core material and there were big problems with corrosion due to water absorption, UV-Radiation, aging etc. [4]. Figure 2.14 below illustrates the Sandwich Structure efficiency.

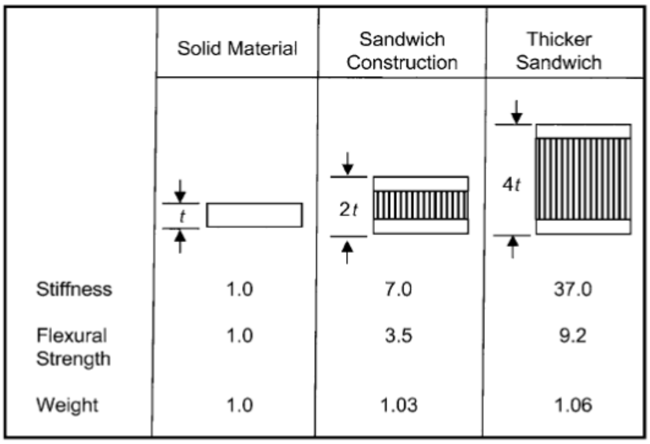

Figure 2.13 Efficiency of sandwich structure [9, p. 256]

However, during the early 60´s, there was a production of different cellular plastics which were suitable as core materials. Soft materials were used in the beginning because of their insulation properties such as polystyrene and polyurethane. Later sandwich structure became very useful and flexible concept as harder cellular plastics with higher densities were possible to produce caused by diverse progresses in material production. [17]

Nowadays, there is an enormous number of different qualities of cellular plastics that are used as core materials. Sandwich panels became an important composite structure in aerospace applications as well as in high performance automobiles, boats, and wind turbines, because it is an extremely lightweight type of construction that exhibits high stiffness and high strength-to-weight ratios. [4]

2.4.1 Sandwich Structure

The importance of studying and analyzing these structures is growing as mentioned in the history of sandwich structure. The calculation for flexural rigidity and other quantities can be seen clearly when considering a sandwich structures as they are a suitable example of composites.

Sandwich structure consists of two stiff, strong faces separated by a lightweight core and joints (adhesives), which can be seen in figure 2.14 below. [4]

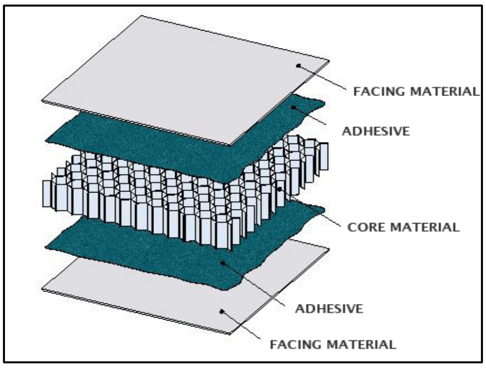

Figure 2.14 Sandwich structure [18, p. 5]

Facing sheets of a typical sandwich structure are mainly thin with a relatively thick lightweight core that separates the two faces in order to increases the moment of inertia. It is for important for engineers or designers of sandwich structures to make the core strong enough to withstand heavy loads and keep their positions. [9]

Each component of the sandwich structures has some specific properties that make the sandwich function effectively as a one-unit object. It is preferred by engineers to make the faces of the sandwich structure from some very well-known metals such as steel, stainless steel, or aluminum due to their stiffness and strength, due to the fact that these materials have some special mechanical properties and they are uncomplicated to use and fabricate. However, in some cases fiber - reinforced plastics are used as face materials as well, because they have good physical properties such as strength. [17]

The core has several critical capacities. It must be sufficiently stiff to keep the separation between the faces steady and it must be so rigid in shear to prevent the faces from sliding over each other. The core must be strong in shear to keep the faces cooperating with each other, if the core is weak in shear, the sandwich loses its stiffness.

To keep the faces and the core cooperating with each other, the adhesive layer must be added as it enables the transmission of the shear forces between the faces and the core. The adhesive must have the capacity to carry shear and tensile stresses. [4]

2.4.2 Flexural Rigidity in Sandwich Beams

Since sandwiches are composites and according to their properties, engineers managed to modify and adjust the beam theories so that they can be applied to the sandwich structures and can be used in analyzing and calculating flexural rigidity of sandwiches. [4]

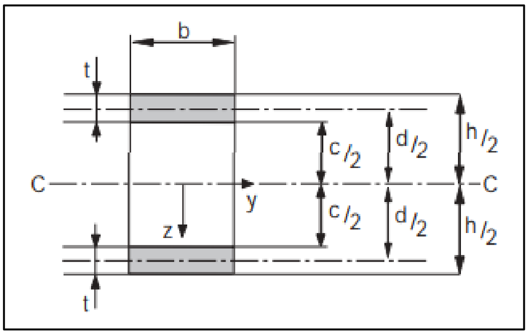

Figure 2.15 Sandwich beam cross section [4]

For beams, the sandwich is slightly different, and it consists of a few different parts where the entire section should be considered when measuring its parts around the centroidal axis. [4] So, the flexural rigidity can be determined as follows:

$$D = E_f \cdot \left(\frac{bt^3}{6}\right) + E_f \cdot \left(\frac{btd^2}{2}\right) + E_c \cdot \left(\frac{bc^3}{12}\right) \qquad (19)\ [4]$$

Where,

E_f = moduli of elasticity of the faces (index f)

E_c = moduli of elasticity of the core (index c)

Equation 22 above consists of three terms that calculates flexural rigidity depending on the local flexural rigidity of the faces about their own centroidal axes, bending about the centroidal axis of the entire cross section and the flexural rigidity of the core about its own centroidal axis. [4]

When this equation is applied to a sandwich structure and comparing the results to the results obtained earlier for a six-section modulus found in page 28-29, it can be seen that the sandwich is a special case of a six-layer parallel axis theorem.

According to some studies and experiments on sandwich structure with thin faces, engineers and scientist concluded that the amount of flexural rigidity due to faces around their own centroidal axis is very small (less than 1%) compared to the rigidity of the bending of the entire cross section around the centroidal axis, therefore it can be neglected especially when:

$$\frac{d}{t} > 5.77$$

And the proportion is less than 0.25% at a ratio of:

$$\frac{d}{t} > 11.55$$

Since the faces are thin, not only the first term can be ignored but in fact experiments showed that the third term of equation 28 is also very small (less than 1%) of the second term which means that it can be ignored as well when:

$$\left(\frac{E_f}{E_c}\right) \cdot \left(\frac{td^2}{c^2}\right) > 16.7 \qquad (20)\ [4]$$

So, the formula for flexural rigidity D is then reduced to:

$$D = E_f \cdot \left(\frac{btd^2}{2}\right) \qquad (21)\ [4]$$

As a mentioned earlier this is almost the same case as six-section module and when applying the parallel axis theorem, one would expect the same calculations and results. So, the expectation is that the rigidity at the outer layers contributes with a higher amount of flexural rigidity than the inner layers when considering the entire sandwich structure.

2.4.3 Deflection of Beams

This section deals with beam deflection. Deflection occurs when a specified load is applied to a cross section beam, and the amount of deflection and stress are very important to determine for sandwich structures and beams, for example in case of sandwich, as the core has a relatively low shear modulus, and the beam may deflect a considerable amount due to shear deformations, to neglect the proper consideration of the shear modulus of the core may lead to unconservative prediction of deflections or critical loads. [11]

In addition, it is very important to know the limits of deflections for sandwich structure beams to ensure the integrity and stability of a structure or to prevent any attached brittle materials from cracking Therefore, it is essential to determine the modulus and take its effect into account. [19]

The deflection of beams depends mostly on the modulus of elasticity of the chosen material and must occur within the elastic limit of the material. They are determined using the elastic theory. There is always some additional deflection in the material due to shear, but it is normally so small that it can be neglected. Figure below illustrates the deflection and curvature due to bending. [11]

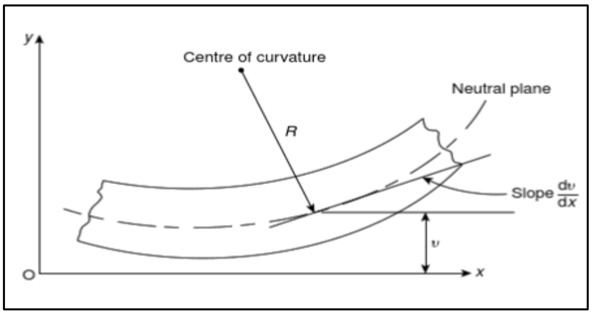

Figure 2.16 Deflection and curvature due to bending [20]

There are different types of supports for example supports that resist a force like a pin or a displacement, another supports for instance resist a moment like a fixed end support, resist displacement or a rotation [20]. The following examples illustrate the deflection of a cantilever and a simply supported beam. [21, pp. 33-35]

- Cantilever beam:

A cantilever beam is when a beam is attached at only one end and free on the other end as shown in figure below. [21, pp. 33-35]

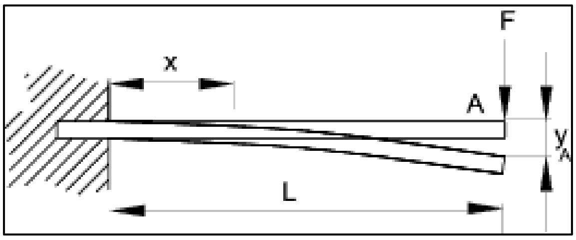

Figure 2.17 Cantilever beam [20]

The deflection for a cantilever beam at any section in terms of x at free end is given by:

$$Y = \frac{Fx^2}{6EI}(3L - x) \tag{22}$$

And the maximum deflection at the free end is:

$$Y_{max} = \frac{FL^3}{3EI} \tag{23}$$

When the load F is concentrated at any point as shown below, then the deflection is:

$$Y = \frac{Fx^2}{6EI}(3a - x) \quad \text{for } 0 < x < a \tag{24}$$

$$Y = \frac{Fa^2}{6EI}(3x - a) \quad \text{for } a < x < L$$

And the maximum deflection in this case is:

$$Y_{max} = \frac{Fa^2}{3EI}(3L - a) \tag{25}$$

When the load is constant and uniformly distributed (W), then the deflection is:

$$Y = \frac{\mathbf{W}x^2}{24EI}(x^2 + 6L^2 - 4Lx) \tag{26}$$

And the maximum deflection in this case is:

$$Y_{max} = \frac{WL^4}{8EI} \tag{27}$$

When the load is variable but still uniformly distributed then the deflection is:

$$Y = \frac{W_0x^2}{120LEI}(10L^3 - 10L^2x + 5Lx^2 - x^3) \tag{28}$$

Similarly, the maximum deflection in this case is:

$$Y_{max} = \frac{W_0L^4}{30EI} \tag{29}$$

For the couple moment M at the free end of the beam, we have:

$$Y = \frac{Mx^2}{2EI} \tag{30}$$

And maximum is:

$$Y_{max} = \frac{ML^2}{2EI} \tag{31}$$

Where,

Y = deflection (mm)

x = d = distance (m)

F = force (N)

L = length (m)

I = area moment of inertia (m^4)

E = bending modulus (material property) (Pa)

M = bending moment (Nm)

W = distributed load (N/m)

❖ Simply supported beam:

A simply supported beam has pinned support at one end and a roller support at the other end. In figure 2.17 below represents the pinned end and B the roller end. [21, pp. 33-35]

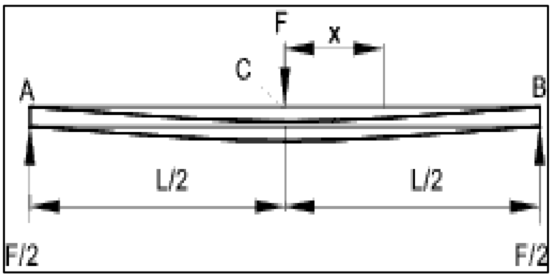

Figure 2.18 Simply supported beam [20]

The governing equations of this module are very similar to the previous case when it comes to types of loads, location of the loads, etc. A review and brief description of the equations are given and illustrated in figure below.

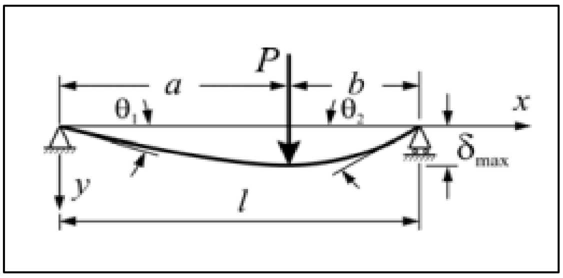

Figure 2.19 Simply supported beam, deflection at center

Deflection at concentrated load (P = F) at the center is:

$$Y = \frac{Fx}{12EI}\left[\frac{3L^2}{4} - x\right] \text{ for } 0 < x < \frac{1}{2} \quad (32)\ [20]$$

And the maximum deflection ($\delta_{max} = y_{max}$) at the center is:

$$Y_{max} = \frac{FL^3}{48EI} \quad (33)\ [20]$$

Concentrated load F at any point:

$$Y = \frac{Fbx}{6LEI}(L^2 - x^2 - b^2) \text{ for } 0 < x < a \quad (34)$$

$$Y = \frac{Fb}{6LEI}\left(\frac{L}{b}(x-a)^3 + (L^2 - b^2)x - x^2\right) \text{ for } a < x < L$$

Maximum deflection at the center:

$$Y_{max} = \frac{Fb(L^2 - b^2)^{3/2}}{9\sqrt{3LEI}} \quad \text{at } x = \sqrt{(L^2 - b^2)/^3} \quad (35)$$

$$Y = \frac{Fb}{48EI}(3L^2 - 4b^2) \quad \text{at the center if } a > b$$

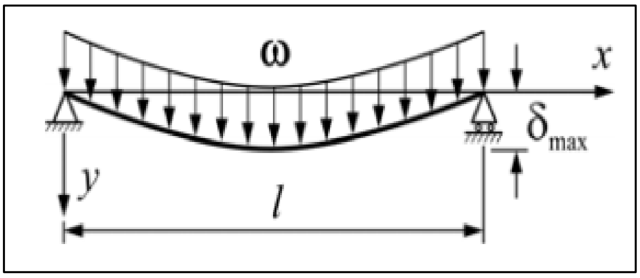

Figure 2.20 Simply supported beam, uniformly distributed load

For a uniformly distributed load (ω = W)N/m as shown in figure 2.20 above, then equations are:

$$Y = \frac{Wx}{24EI}(L^3 - 2Lx^2 - x^3) \quad (36)$$

Maximum deflection:

$$Y_{max} = \frac{5WL^4}{384EI} \tag{37}$$

When the load is uniform and variable, then deflection can be expressed as:

$$Y = \frac{W_0 x}{360LEI}(7L^4 - 10L^2x^2 + 3x^4) \tag{38}$$

When the couple moment M is at the right end, then:

$$Y = \frac{MLx}{6EI}\left(1 - \frac{x^2}{L^2}\right) \tag{39}$$

And the maximum is:

$$Y_{max} = \frac{ML^2}{9\sqrt{3EI}} \text{ at } x = 1/\sqrt{3} \tag{40}$$

Where,

Y = deflection (mm)

x = d = distance (m)

F = force (N)

L = length (m)

I = area moment of inertia (m^4)

E = bending modulus (material property) (Pa)

M = bending moment (Nm)

b = width (m)

W = distributed load (N/m)

3 METHOD

For the practical part and laboratory experiments of this thesis work, the following module, software, and concepts are considered.

1. Composite Compressive Strength Modeler (CCSM)
2. A case of a solid UD lamina
3. Sandwich structure
4. Three-Point and Four-Point bending
5. Data analysis

3.1 Composite Compressive Strength Modeler (CCSM)

According to User´s Manual of CCSM modeler, it deals with many different aspects of beams and plates in terms of deflection, flexural rigidity, etc. Some of the main features of this software include classical laminate theory, stress-strain analysis, failure prediction for composite plates, and a very useful user-expandable database to store material and geometrical properties. However, in this thesis work the modeler is used for determining the stiffness and compliance matrices of beams when sufficient information about the beams, lamina, stiffness, etc. are given. [22]

In this thesis work when stiffness and compliance matrices are considered using CCSM modeler the flexural rigidity D will be calculated using width (b) and d_{11}. And in this case, it is:

$$D_{theory} = \frac{b}{d_{11}} \quad (41)\ [22]$$

3.1.1 A Solid UD Lamina

A lamina is a building block of modern composites laminated structures, a lamina is also known as a ply or a layer and in our case the UD lamina refers to a unidirectional lamina since each lamina may have more than one type of fibers and these fibers may be oriented in different directions, different thickness, fiber orientation angle and matrix material. [22]

The geometry of a laminate is normally layers that have three planes of material that are symmetric. Therefore, they exhibit orthotropic behavior (having three mutually perpendicular planes of elastic symmetry at each point).

When a lamina cut cross these planes of symmetry, they will exhibit the same mechanical properties. Lines which are normal to these planes of material symmetry are called material axes which are designed as 1, 2, and 3 they are also known as principal material directions. Figure 3.1 below illustrate these axes in a UD lamina. [22]

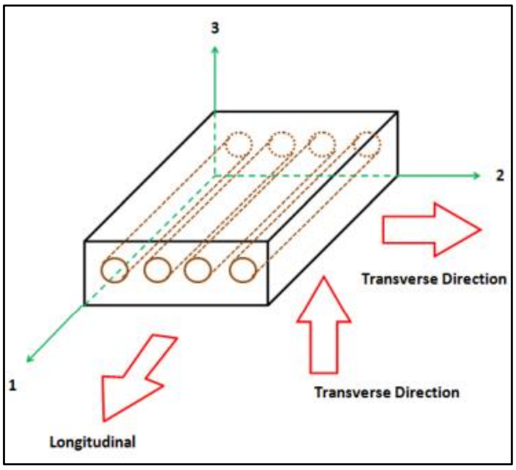

Figure 3.1 Principal material axes in a UD lamina [24]

When dealing with fibers, matrix and lamina, it should be mentioned that fibers´ strength and stiffness are significantly larger than that of the matrix, a lamina is stiffest and strongest in longitudinal direction which is the 0-degree direction and they are very weak in the 90-degree direction because the load must be carried by the much weaker polymeric matrix.

A lamina´s mechanical properties in any direction lying in the 2-3 planes are quite similar and therefore, a unidirectional lamina is considered as transversely isotropic. Each layer has approximately the same properties in-plane but different properties through the thickness. [23]

3.2 Bending

Bend testing also known as flex or flexural testing is commonly performed to measure the flexural strength and modulus of all types of materials and products. The bending of a material or product can be tested using a bending machining tool. [24]

A universal bending machine consists of a basic machine that can be adjusted and used for a variety of bends. The basic machine consists of a CNC-operated (computer numerically controlled), a work bench and a software for programming and operating. There are three key analysis when performing bend testing which are, the flexural modulus that deals with measuring slope, stress-strain curve, and stiffness of materials, flexural strength which measures the maximum force that a material can resist before it breaks or yields, and yield point of a material, it is a point at which the material can´t restore its normal shape after it. [24]

3.2.1 Three-Point & Four-Point Bending

Flexural rigidity, modulus of elasticity and other related quantities can be determined when the values of Three-Point bending are given. The advantage of using three-point flexural test is the ease of the specimen preparation and testing. Therefore, the three-point bending will be the module in the practical part of this thesis work. [25]

The four-point bending flexural test is very similar to the three-point bending test. The major difference is the addition of a 4^{th} bearing, which brings a much larger portion of the beam to the maximum stress. This difference needs to be taken into account when studying for example the brittle materials, since it can be used to indicate the flexural strength and crack initiation for instance, in case of asphalt mixtures that are used in road paving, however the four-point bending test won´t be included in the practical part of this scientific discourse. [26]

Tables below show some international standard testing methods.

Table 2. ASTM standards for three-point bend. [27]

ASTM	Standard Test Method
ASTM-C1161	Flexural Strength of Advanced Ceramics at Ambient Temperature
ASTM-C1341	Flexural Properties of Continuous Fiber-Reinforced Advanced Ceramic Composites
ASTM-C1684	Flexural Strength of Advanced Ceramics at Ambient Temperature-Cylindrical Rod Strength
ASTM-C203	Breaking Load and Flexural Properties of Block-Type Thermal Insulation
ASTM-C473	Physical Testing of Gypsum Panel Products
ASTM-C598	Annealing Point and Strain Point of Glass by Beam Bending
ASTM-C674	for Flexural Properties of Ceramic White-Ware Materials
ASTM-D1184	form Flexural Strength of Adhesive Bonded Laminated Assemblies
ASTM-D143	Small Clear Specimens of Timber
ASTM-D2344	for Short-Beam Strength of Polymer Matrix Composite Materials and Their Laminates
ASTM-D3044	Shear Modulus of Wood-Based Structural Panels
ASTM-D349	Laminated Round Rods Used for Electrical Insulation
ASTM-D4476	Flexural Properties of Fiber Reinforced Pultruded Plastic Rods
ASTM-D7264	Flexural Properties of Polymer Matrix Composite Materials
ASTM-D790	Flexural Properties of Unreinforced and Reinforced Plastics and Electrical Insulating Materials
ASTM-E855	Bend Testing of Metallic Flat Materials for Spring Applications Involving Static Loading
ASTM-F1575	Determining Bending Yield Moment of Nails
ASTM-F2193-02	Components Used in the Surgical Fixation of the Spinal Skeletal System

Table 3. ASTM standards for four-point bend. [27]

ASTM	Standard Test Method
ASTM-C1161	Flexural Strength of Advanced Ceramics at Ambient Temperature
ASTM-C1341	Flexural Properties of Continuous Fiber-Reinforced Advanced Ceramic Composites
ASTM-C1368	Determination of Slow Crack Growth Parameters of Advanced Ceramics by Constant Stress-Rate Strength Testing at Ambient Temperature
ASTM-C1576	Determination of Slow Crack Growth Parameters of Advanced Ceramics by Constant Stress Flexural Testing (Stress Rupture) at Ambient Temperature
ASTM-C158	Strength of Glass Flexure (Determination of Modulus of Rupture)
ASTM-C1674	Flexural Strength of Advanced Ceramics with Engineered Porosity (Honeycomb Cellular Channels) at Ambient Temperatures
ASTM-C1684	Flexural Strength of Advanced Ceramics at Ambient Temperature-Cylindrical Rod Strength
ASTM-C393	Core Shear Properties of Sandwich Constructions by Beam Flexure
ASTM-C480	Flexure Creep of Sandwich Constructions
ASTM-C651	Flexural Strength of Manufactured Carbon and Graphite Articles Using Four-Point Loading at Room Temperature
ASTM-D6272 –	Flexural Properties of Unreinforced and Reinforced Plastics and Electrical Insulating Materials by Four-Point
ASTM-D7249	Facing Properties of Sandwich Constructions by Long Beam Flexure
ASTM-D7264	Flexural Properties of Polymer Matrix Composite Materials
ASTM-D790	Flexural Properties of Unreinforced and Reinforced Plastics and Electrical Insulating Materials
ASTM-E855	Bend Testing of Metallic Flat Materials for Spring Applications Involving Static Loading

4 RESULTS

This section deals with the results, graphs, CCSM modeler, experiments, and data analysis in both theoretical and experimental cases as below.

4.1 Bending Test and CCSM Modelling

In this section the goal is to make a comparative analysis between theoretical and experimental flexural rigidity D on a solid fiberglass and a sandwich beam with different orientations using the material testing machine under three-point bending with a uniform constant load and the CCSM modeler for the theoretical part.

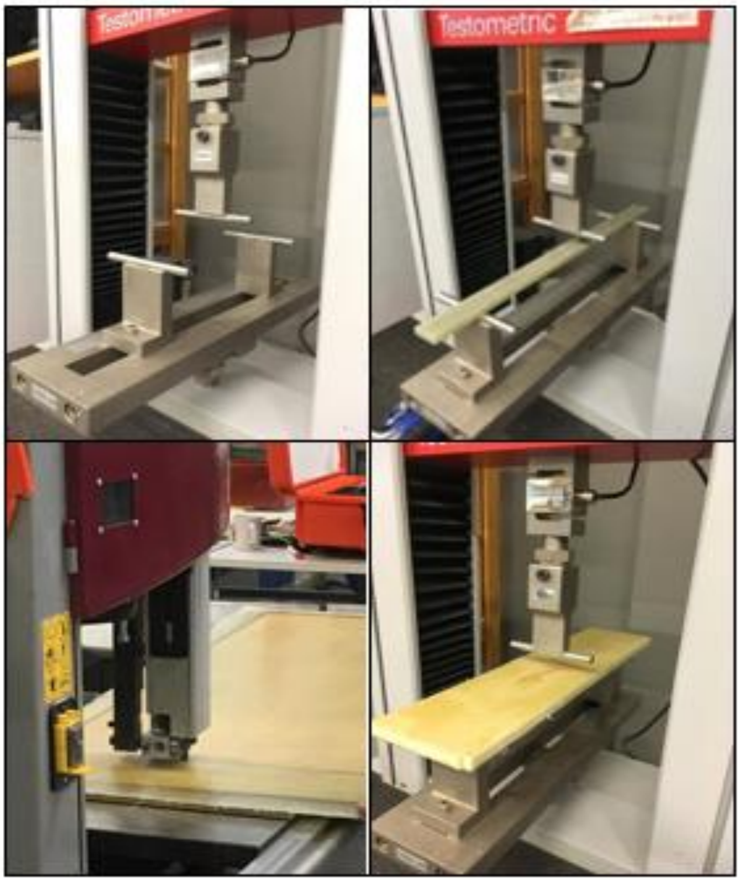

Figure 4.1 Materials testing machine

A solid beam of 10 plies and a sandwich beam of 4 plies are used as examples to obtain the theoretical results of stiffness and compliance matrices for zero degree [0º] using the software package CCSM modeler.

The CCSM modeler program can be used by entering the obtained geometry of the beam, that is, the required beam dimensions, save ply data in the input and then calculate for elastic properties according to the user´s manual.

It should be mentioned that for the sake of time saving and ease of use the same lamina properties and thickness can be saved and used for other layers as long as they are of the same material. However, when the lamina is of different material then the reentering data of each lamina along with corresponding properties is required.

1. Theoretical test solid fiberglass (CCSM modeler):

The module in this example as mentioned earlier is symmetric and all layers are made of same material (fiberglass). Both theoretical and experimental tests are applied to this module according to the values given in the table below.

Table 4. Values for the CCSM Modeler.

Property	Symbol	Unit
Width	b	25 mm
Thickness (unidirection)	t_A	0.75 mm
Longitudinal modulus	E_{11}	36.5 GPa
Transverse direction	E_{22}	5.7 GPA
Poisson number	Nu	0.3
Shear Modulus	G_{12}	2.1GPa
Angle	℃	0

The first step is to enter the above data to the CCSM modeler.

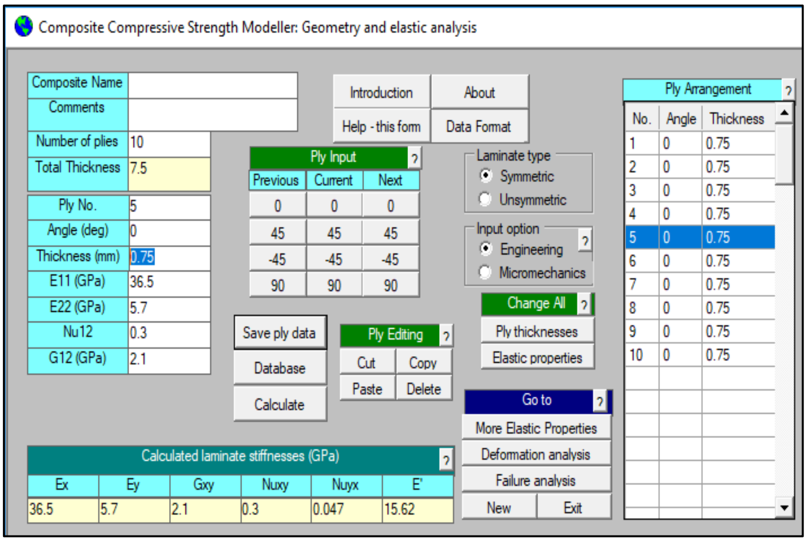

Figure 4.2 Solid fiberglass - theoretical

The data used in this example are taken from a real lamina tested in the laboratory using a microscope. So, the data are almost exact. After running the software, the following results are obtained, and we are mainly interested in the d_{11} value that will be used in determining the value of flexural rigidity.

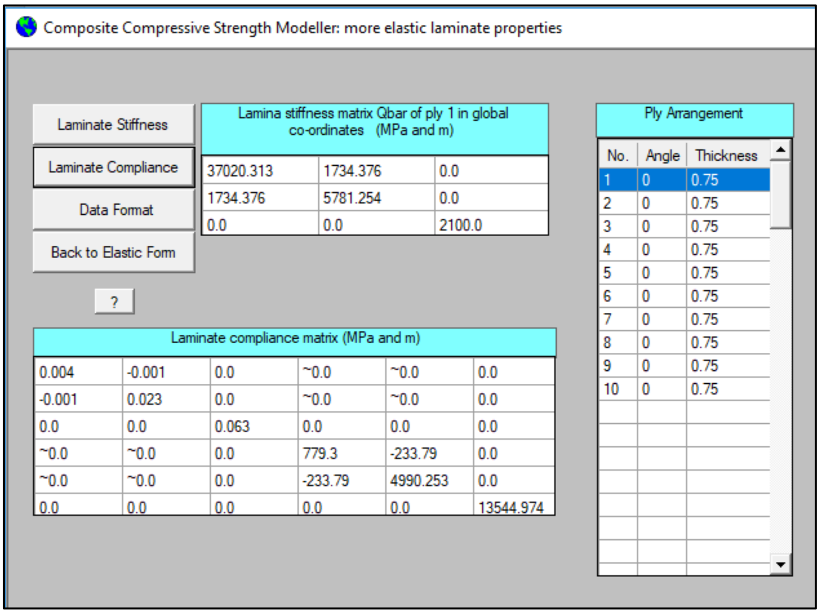

Figure 4.3 Solid fiberglass – theoretical – d_{11}

Based on the CCSM modeler the value of d_{11} and knowing the total thickness of the lamina, that is:

b = 25 mm

$d_{11} = 779.3$

So, the theoretical flexural rigidity is: [22]

$$D_{theory} = \frac{b(mm)}{d_{11}(Nm)^{-1}} = \frac{25\ Nmm^2}{779.3} \cdot 1000 \ = 32\ Nmm^2$$

The flexural rigidity D_{theory} of the 10 layers fiber glass plies is according to the result above is 32 Nmm^2.

2. Experimental test solid fiberglass:

This test is performed in the laboratory using material testing machine model M 350-3CT available at Arcada laboratory. The same exact solid beam along with the same exact data are used for this experimental test for the sake of comparison between the theoretical and experimental tests and determining the effectiveness and accuracy of CCSM modeler. Figure below shows the linear elasticity and the slope which is of main interest in calculating flexural rigidity.

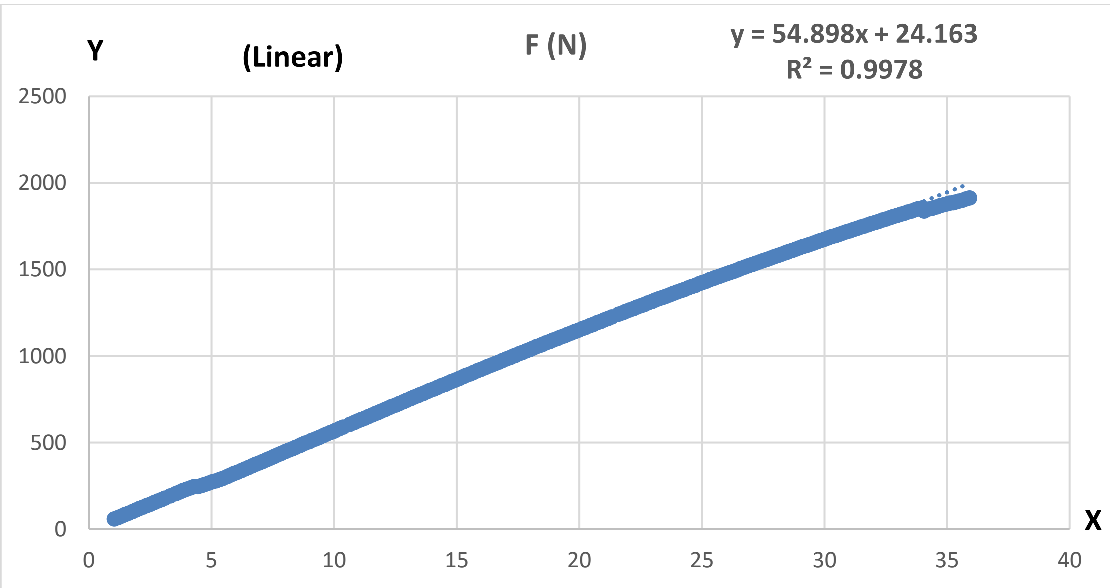

Figure 4.4 Solid fiberglass - experimental

The experimental calculations of flexural rigidity D are as follows:

$$\text{Slope} = \frac{dF}{dy} = k = \frac{48D}{L^3} \qquad (42)\ [22]$$

Where,

L = length between supports

$k = \text{Slope} = 54.898\,\frac{N}{mm}$

Rearranging the equation 50 leads to:

$$D_{exp} = k\frac{L^3}{48} = k\ \frac{(300mm)^3}{48}$$

$$D_{exp} = 54.898 \cdot\ 562500\ \left[\frac{N}{mm}\right]\left[\frac{mm^3}{1}\right]$$

$$=\ 30880125\ Nmm^2$$

$$D_{exp} \approx 31\ Nmm^2$$

The relative error is $1 - \frac{D_{exp}}{D_{theory}} = 1 - \frac{31}{32} = 1 -\ 0{,}96875$, corresponding to 3.1%

1. Theoretical test sandwich beam (number 1):

This example is a sandwich beam with four plies and width of 13mm total, with a 1.5mm width for each face and the longitudinal modulus in this example is assumed 15.5 GPa based on the obtained experimental results of this example as shown in figure 4.5 below.

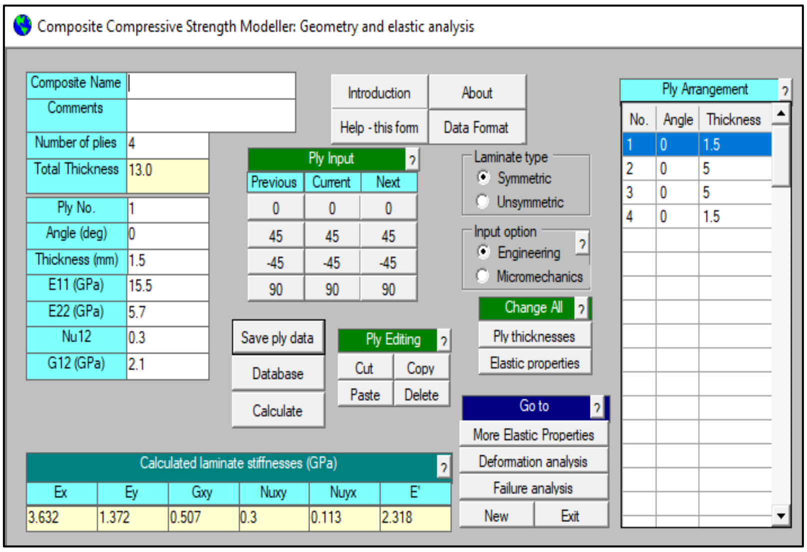

Figure 4.5 Sandwich beam 1 - theoretical

After entering the required data and running the simulation, the following results are obtained as shown below.

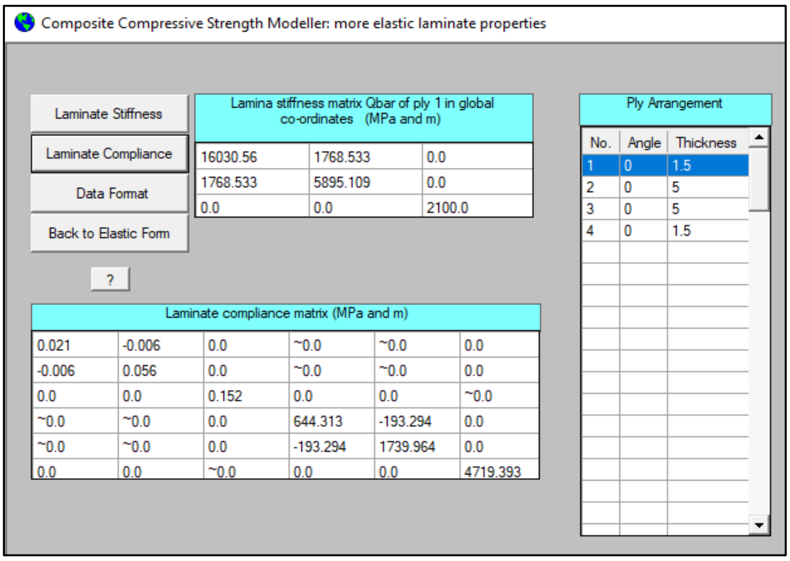

Figure 4.6 Sandwich beam 1 – theoretical – d_{11}

In a similar way to the first case, the theoretical flexural rigidity can be calculated as follows:

$$D_{\text{theory}} = \frac{b(\text{mm})}{d_{11}(\text{Nm})^{-1}} = \frac{100\ \text{Nmm}^2}{644.313} 1000 = 155\ \text{Nmm}^2$$

2. Experimental test sandwich beam (number 1):

In this laboratory test a sandwich beam made of core cell M80 and the facings made of vinyl ester is used and the other material properties and constants are taken from table 5 page 56.

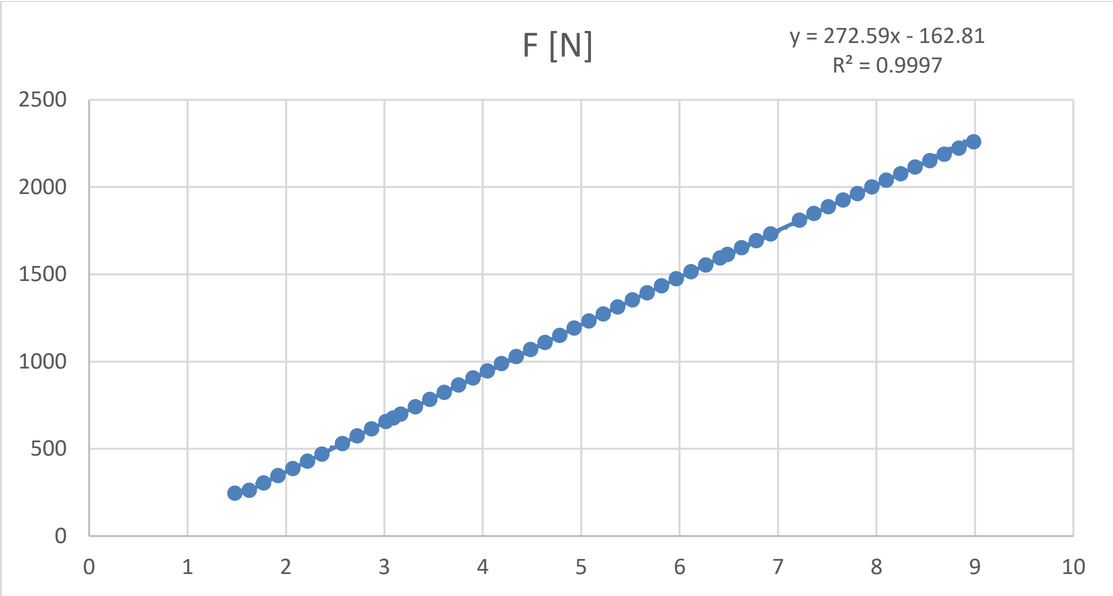

Figure 4.7 Sandwich beam 1 - experimental

Applying the flexural rigidity equation again with the obtained slope k, then:

$$D_{exp} = k\frac{L^3}{48} = k\ \frac{(300\text{mm})^3}{48}$$

$$D_{exp} = 272.59 \cdot\ 562500\ \left[\frac{N}{mm}\right]\left[\frac{mm^3}{1}\right]$$

$$=\ 153331875\ \text{Nmm}^2$$

$$D_{exp} = 153{,}3\ \text{Nmm}^2$$

The relative error is $1 - \frac{D_{exp}}{D_{theory}} = 1 - \frac{153{,}3}{155} = 1 - 0{,}989032258$, corresponding to 1,1%

1. Theoretical test sandwich beam (number 2):

The sandwich beam in this test differs from the previous one by angle of orientation, otherwise all the material, data, dimensions are the same. Figure below shows a screenshot of CCSM modeler with entered data. For convenience and suitability, the longitudinal modulus E_{11} is taken 50 GPa in this test in order to meet the results obtained from the experimental part, since the slope of the linear elasticity gets much smaller in this case due to orientation of the sandwich beam under three-point bending.

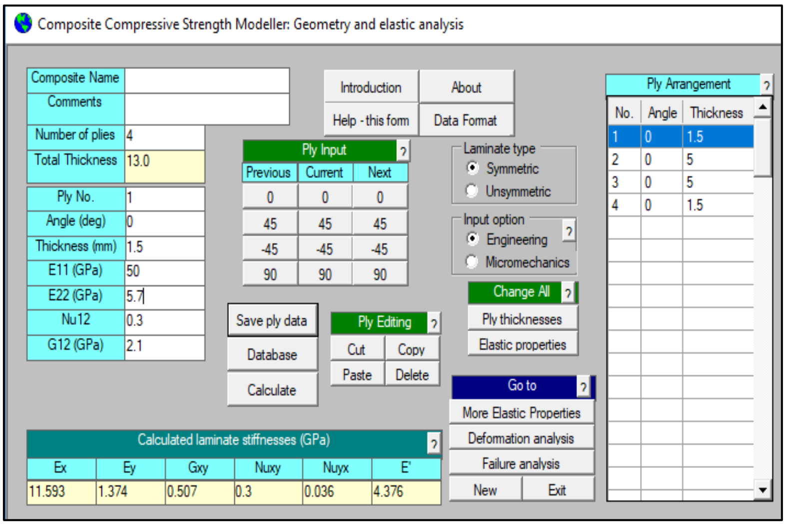

Figure 4.8 Sandwich beam 2 - theoretical

After running CCSM modeler software, then the following results are obtained for compliance matrix d_{11}.

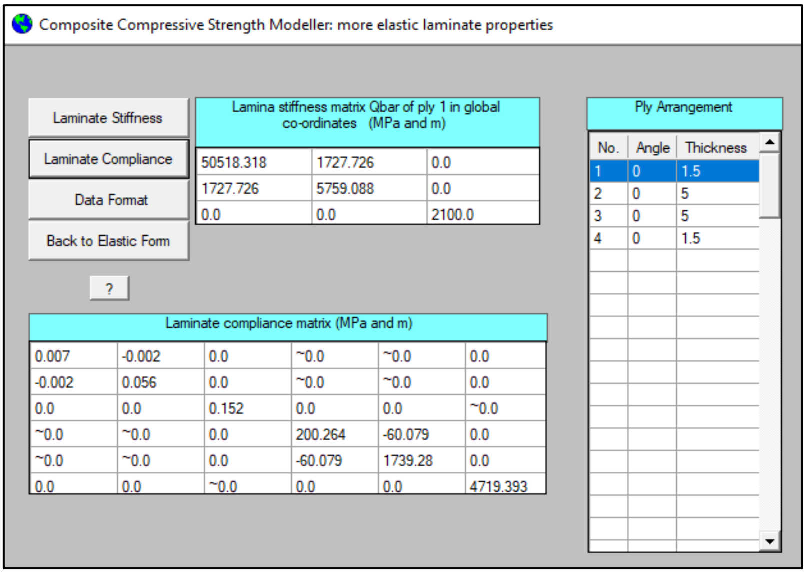

Figure 4.9 Sandwich beam 2 – theoretical d_{11}

In the same way the flexural rigidity D is calculated bellow:

$d_{11} = 200.264$

$$D_{theory} = \frac{b(mm)}{d_{11}(Nm)^{-1}} = \frac{90 \ Nmm^2}{200.264} 1000 = 45.1 \ Nmm^2$$

The theoretical flexural rigidity D is about $45.1 \ MNmm^2$ which will be discussed and compared with the practical results in details in conclusion section.

2. Experimental test for sandwich beam (number 2):

For the sake of comparison, an experimental test is performed at the laboratory for the same sandwich beam and the following figure and calculations are obtained.

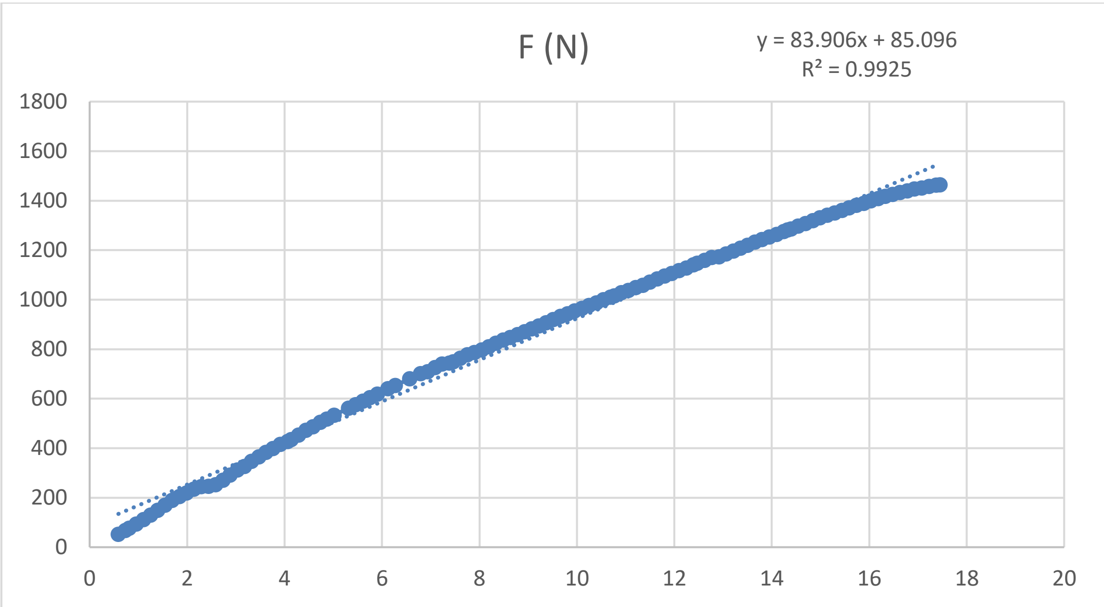

Figure 4.10 Sandwich beam 2 - experimental

Applying the flexural rigidity equation with L = 300mm which is the length between the two supporters and slope k from the linear equation in figure above, then:

$$\mathrm{D_{exp}} = \mathrm{k}\frac{\mathrm{L}^3}{48} = \mathrm{k}\ \frac{(300\mathrm{mm})^3}{48}$$

$$\mathrm{D_{exp}} = 83.906 \cdot\ 562500\ \left[\frac{\mathrm{N}}{\mathrm{mm}}\right]\left[\frac{\mathrm{mm}^3}{1}\right]$$

$$=\ 47197125\ \mathrm{Nmm}^2$$

$$\mathrm{D_{exp}} = 47.2\ \mathrm{Nmm}^2$$

The relative error is $1 - \frac{\mathrm{D_{exp}}}{\mathrm{D_{theory}}} = 1 - \frac{47.2}{45.1} = 1 -\ 1.04653193$, corresponding to -4.6%.

The theoretical and experimental results differ by around 1.1, 3.2 and -4.6 percent in these tests, due to the fact that the beam measurements were not accurate 100 percent during the laboratory test because of the roughness and unleveled edges of the sandwich beam.

5 DISCUSSION

In this scientific discourse, three laboratory experiments were conducted for practical tests along with the use of composite compressive strength modeler (CCSM) for theoretical tests for the sake of comparison between the results obtained in case of a solid fiberglass and a sandwich beam.

The experiments were conducted using materials testing machine in order to have an understanding of the behavior of the beams under three-point bending. The properties of the material used in these tests were provided by the manufacturer (Gurit company). The tests were conducted in longitudinal direction in order to obtain relevant constitutive behavior of the facing material.

For the theoretical part, from different available software packages, the CCSM modeler software was used for its simplicity, friendly user interface and its fast performance. Moreover, for simple geometries it may be clear what the loading is. However, for more complicated geometries the program may be used as part of a larger calculation to find d_{11} which can be used to calculate flexural rigidity D. As well it can be used to check for failure at critical points in the structure.

Both experimental and theoretical results obtained along with a percentage comparison between both cases are presented in the table below.

Table 5. Results

Specimen and material	C (mm)	b (mm)	L (mm)	Slope k(N/mm)	Experimental D($MNmm^2$)	Theoretical D($MNmm^2$)	Comparison percentage
Solid (fiberglass)	7.5	25	300	55.284	31.1	32	3.1%
Sandwich beam 1	13.0	100	300	272.59	153.3	155	1.1%
Sandwich beam 2	13.0	90	300	86.888	47.22	45.1	−4.6%

Where,

L = Distance between the supports (m)

b = width (m)

C = thickness (m)

k = slope (N/mm)

D = flexural rigidity (Nm^2)

From the results one can see that, the theoretical results obtained using CCSM modeler are very close to the experimental results and in all cases the percentage difference ranged from 1.1 percent to 4 percent which is acceptable and reasonable. The results also show the effectiveness and accuracy of the CCSM modeler.

For the experimental tests and under the same load and speed, it appeared that the direction of orientation plays a major role in the stiffness of beams as it can be seen from the flexural rigidity values. The composite seemed to be stronger along the direction of orientation of the fiber with flexural rigidity value of 153.3 Nmm^2 and weaker when the direction was perpendicular to the fiber with flexural rigidity of 47.22 Nmm^2.

Results also showed that the beam displayed a linear behavior to the cracking moment in both cases. The sandwich beam carried most of the bending and in-plane loads in the facings, while the core was the main source of flexural stiffness, out-of-plane shear, and compressive behavior.

The solid fiberglass was weaker as expected, its flexural rigidity D=31.1 $MNmm^2$ was much less than that for sandwich beams of the same length L between the two supporters and under the same load and the rest of the conditions.

6 CONCLUSION

The basic idea of this thesis work is to investigate and determine the flexural rigidity that is, the bending stiffness of beams with different cross sections of the same material properties. Three beams were tested under three-point bending using material testing machine and the other purpose of this thesis work was to compare the results obtained in both cases that is, composite compressive strength modeler (CCSM) results and laboratory results.

The conclusions of this thesis work are based on the sandwich beam theory and experiments done in the laboratory as well on the CCSM modeler software. One of the main conclusions is that the sandwich constructions are very suitable in engineering structures and many other fields due to their lightweight structures and their flexural stiffness along with manufacturing of new composites with high qualities regarding bending, strength, and many other properties.

From Euler-Bernoulli beam theory concept, it is essential to mention that the second moment of inertia of cross-sectional shapes of beams play one of the biggest roles and in some cases even the main key role. It is easy to see that a beam with a higher moment of inertia is more resistance to bending than a beam with a smaller moment of inertia according to the results obtained. Therefore, it is logical to have a cross sectional area concentrated away from the beam center. As a general rule, to increase rigidity of a beam, it is recommended to make the second moment of area as large as possible.

One more important conclusion achieved in this thesis work is that the structural performance of the sandwich beam doesn´t only depend on the properties of the skin, but also on orientation and geometrical dimensions of the component, that is why it is essential to choose the right orientation, geometrical dimensions, and other factors into considerations to make sure the designed structure satisfies specific strength, stability, and deflection requirements. Therefore, the bending stiffness analysis plays a big role in choosing materials, as it shows how the material behaves under certain loads that helps engineers in building, designing, manufacturing and in many other fields.

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